

# Gauge Theories from Dp-branes

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In this talk we discuss the need to introduce a string theory in order to obtain a consistent quantum theory of gravity unified with gauge interactions. We then discuss some basic properties of string theory and the origin and the properties of the D(irichlet)-branes. Finally we use them for discussing the Maldacena conjecture and its extension to non-conformal and less supersymmetric theories.

## 1. BEYOND THE STANDARD MODEL

Strong, weak and electromagnetic interactions are described by the standard model that is a gauge field theory based on the group  $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ . It has three gauge coupling constants  $g_1, g_2$  and  $g_3$  and the gauge fields are the 8 gluons,  $W^\pm, Z^0$  and the photon. It contains two scales: the QCD scale  $\Lambda_{QCD} \sim 250 \text{ MeV}$  corresponding to the dimension of a proton that is about  $10^{-13} \text{ cm} = 1 \text{ Fermi}$  and the Fermi scale  $\sim 250 \text{ GeV}$  corresponding to the scale at the which the gauge group  $SU(2)_L \otimes U(1)_Y$  is broken into  $U(1)_{em}$ . All present experimental data fully agree with high precision with the predictions of the standard model. As a consequence we can at the moment only speculate on what will happen at higher energy and on which additional scales we could expect. Since the running of the coupling constants  $g_i$  is entirely predictable from the low-energy particle spectrum and quantum numbers with respect to the gauge groups of the standard model, one can ask oneself if they have the tendency to get together at some higher energy. It turns out that indeed they do at an energy of the order of  $M_{GUT} = 10^{16} \text{ GeV}$  and this suggests that the three groups of the standard models may get unified at such a high energy [1]. In addition we know that gravity becomes strong at the Planck mass given by:

$$M_{Pl} \equiv \sqrt{\frac{\hbar c}{G_N}} = 1.2 \cdot 10^{19} \text{ GeV} \quad (1)$$

This means that the standard model, although renormalizable, cannot be a fundamental theory valid at all energies. It is only an effective theory valid at scales  $\ll M_{GUT}, M_{Pl}$ . But, if this is the case, then we get the hierarchy problem because we expect a Higgs particle with a mass of the order of the cut-off corresponding in our case to  $M_{GUT}$  or  $M_{Pl}$ , while we need a Higgs particle with a mass of the order of the Fermi scale  $\ll M_{GUT}$  in order to break  $SU(2)_L \otimes U(1)_Y$  of the standard model in  $U(1)_{em}$ . The most popular way out of this problem is to extend the standard model to the minimal supersymmetric standard model where for each particle of the standard model we include also its supersymmetric partners that are required to have a mass of the order of the Fermi scale. Actually it turns out that in this case the three running coupling constants meet all at the same point corresponding to an energy equal to  $2 \cdot 10^{16} \text{ GeV}$  [1]. But also the supersymmetric standard model cannot be a fundamental theory because it does not incorporate quantum gravity and we know that when we reach the Planck mass a classical description of gravity is not anymore consistent. This follows from the fact that a quantum field theory of gravity is not renormalizable. In fact a theory based on pointlike constituents as the case of a field theory has already at the classical level problems due to the short-distance or ultraviolet divergences. These divergences are in fact already present at the classical level in electrodynamics [2]. That is why one introduces the classical electron radius

that is just an ultraviolet cut-off given by:

$$\frac{e^2}{r_0} = mc^2 \rightarrow r_0 = \alpha \frac{\hbar}{mc} = \frac{1}{137} \frac{\hbar}{mc} \quad (2)$$

where  $\alpha$  is the fine structure constant of electromagnetism. In the quantum theory some of those divergences in general survive and we must renormalize the theory in order to compare with the experiments. But this requires the quantum theory to be renormalizable. Since this is not the case for quantum gravity then it is natural, in order to construct a quantum theory of gravity, to go away from the pointlike structure. The simplest case is that of a one-dimensional string. It turns out that a string theory is able to provide us a consistent quantum theory of gravity unified with gauge theories. Some basic elements of string theory will be discussed in the next section.

## 2. STRING THEORY

The action of the bosonic string can be constructed in analogy with that of a spinless particle. The motion of a spinless particle is described by its coordinate  $x^\mu(\tau)$  in flat Minkowski space as a function of an arbitrary parameter  $\tau$ . Analogously the motion of the bosonic string is described by its coordinate  $x^\mu(\tau, \sigma)$  as a function of two arbitrary parameters  $\sigma$  and  $\tau$ . As the action for a free spinless particle is proportional to the length of its world-line, so the action of the bosonic string will be proportional to the area of its world-sheet:

$$-mc \int \sqrt{-dx_\mu dx^\mu} \Rightarrow -Tc \int \sqrt{-d\sigma_\mu d\sigma^\mu} \quad (3)$$

where  $m$  is the particle mass and  $T$  is the string tension having dimension of an energy per unit length. The fact that the actions in eq.(3) are proportional to geometrical objects implies that the variables  $\tau$  and  $\sigma$  can be reparametrized at will without changing the physics. This is reflected in the fact that both actions in eq.(3) are invariant under reparametrizations of the world-sheet(line) variables. In order to quantize the system is, however, convenient to choose a covariant gauge; the proper-time gauge in the case of the point-particle where  $\tau$  is identified with the proper-time of the

particle and the orthonormal gauge characterized by the conditions

$$\dot{x}^2 + (x')^2 = \dot{x} \cdot x' = 0 \quad (4)$$

in the case of the bosonic string. In these gauges the previous actions become quadratic in the coordinate  $x^\mu$ :

$$-\frac{(mc)^2}{2} \int d\tau \dot{x}^2 \Rightarrow \frac{Tc}{2} \int d\sigma d\tau [(x')^2 - \dot{x}^2] \quad (5)$$

where  $\dot{x}$  is the derivative with respect to  $\tau$  and  $x'$  that with respect to  $\sigma$ .

In the orthonormal gauge the string eq. of motion becomes linear and one gets:

$$\left( \frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2} \right) x^\mu(\tau, \sigma) = 0 \quad (6)$$

that must be implemented together with eq.s (4). In addition one gets also the following boundary condition:

$$\frac{\partial x^\mu}{\partial \sigma} \cdot \delta x|_{\sigma=0}^{\sigma=\pi} = 0 \quad (7)$$

where the variable  $\sigma$  has been taken to vary in the interval  $(0, \pi)$ . In the case of a closed string satisfying the periodicity condition  $x^\mu(\tau, \sigma) = x^\mu(\tau, \sigma + \pi)$  one gets the following most general solution of the eq. of motion:

$$\begin{aligned} x^\mu(\tau, \sigma) = & q^\mu + 2\alpha' p^\mu \tau + \\ & + i \frac{\sqrt{2\alpha'}}{2} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[ a_n^\mu e^{-2in(\tau-\sigma)} - a_n^{\dagger\mu} e^{2in(\tau-\sigma)} \right] + \\ & + i \frac{\sqrt{2\alpha'}}{2} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[ \tilde{a}_n^\mu e^{-2in(\tau+\sigma)} - (\tilde{a}_n^\dagger)^\mu e^{2in(\tau+\sigma)} \right] \end{aligned} \quad (8)$$

where  $a_n$  and  $\tilde{a}_n$  are arbitrary parameters. In the case of an open string the boundary condition in eq.(7) can be satisfied by imposing at each string end-point and for each direction one of the two conditions:

$$\frac{\partial x^\mu}{\partial \sigma} = 0 \quad \text{or} \quad \delta x^\mu = 0 \quad (9)$$

The first one corresponds to a Neumann boundary condition and preserves translational invariance, while the second one corresponds to a

Dirichlet boundary condition and violates translational invariance. Imposing a Dirichlet boundary condition corresponds to require that the string end-point is stuck on a  $p$ -dimensional hyperplane called  $p$ -brane, while in the case of a Neumann boundary condition the string end-point is free to move and according to eq.(4) it moves with the speed of light. For this reason a brane on which open strings can have their end-points fixed is called a Dp-brane. We will discuss them later on. Here we give the general solution for an open string only in the case of all Neumann boundary conditions. It is given by:

$$x^\mu(\tau, \sigma) = q^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[ a_n^\mu e^{-in\tau} a_n^{\dagger\mu} e^{in\tau} \right] \cos n\sigma \quad (10)$$

The quantum theory is obtained by imposing the following canonical commutation relations:

$$[a_n^\mu, a_m^{\dagger\nu}] = [\tilde{a}_n^\mu, \tilde{a}_m^{\dagger\nu}] = \delta_{nm} \eta^{\mu\nu}, \quad [q^\mu, p^\nu] = i\eta^{\mu\nu} \quad (11)$$

The spectrum of states of an open string is given by the following expression:

$$\alpha' M^2 = \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n - 1 \quad (12)$$

The lowest string excitation is given by the vacuum state  $|0, k\rangle$  with momentum  $k$  corresponding to a tachyon with mass  $\alpha' M^2 = -1$ , while the next one is a massless abelian vector state  $a_1^{\dagger\mu} |0, k\rangle$ . The gauge vector field can be made non-abelian by introducing Chan-Paton factors at the end-points of the string. The spectrum of the closed bosonic string is given by the following eq.:

$$\frac{\alpha'}{2} M^2 = \sum_{n=1}^{\infty} \left[ n a_n^\dagger \cdot a_n + \sum_{n=1}^{\infty} n \tilde{a}_n^\dagger \cdot \tilde{a}_n \right] - 2 \quad (13)$$

together with the level matching condition

$$\sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n = \sum_{n=1}^{\infty} n \tilde{a}_n^\dagger \cdot \tilde{a}_n \quad (14)$$

The lowest state  $|0, k\rangle$  is again a tachyon with  $\alpha' M^2 = -4$ , while the massless states are described by the state  $a_1^{\dagger\mu} \tilde{a}_1^{\dagger\nu} |0, k\rangle$ . Its symmetric

and traceless part corresponds to the graviton, while its trace and its antisymmetric part correspond respectively to a dilaton and an antisymmetric tensor (two-form potential).

The bosonic string contains not only the particles (gauge bosons and gravitons) that appear in the Standard model and in the Einstein's theory of general relativity, but also their interactions. In fact it can be shown that the low-energy string effective action contains the two terms:

$$S = \int d^D x \sqrt{-G} \left[ -\frac{1}{4g_{YM}^2} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2\kappa^2} R \right] \quad (15)$$

where the Yang-Mills coupling constant and the Newton constant are given in terms of  $\alpha'$  and of the string coupling constant  $g_s$  by

$$g_{YM}^2 \sim g_s (\alpha')^{(D-4)/2} \quad (16)$$

$$2\kappa^2 \equiv 16\pi G_N \sim g_s^2 (\alpha')^{(D-2)/2} \quad (17)$$

The quantum theory of the bosonic string is consistent with Lorentz invariance only if the space-time dimension  $D = 26$ . But what it makes the bosonic string inconsistent is the presence of both the open and closed string tachyons. In order to get rid of them one must go from the bosonic string to the superstring. In this case one introduces additional world-sheet degrees of freedom represented by the world-sheet Dirac field  $\psi^\mu(\tau, \sigma)$  corresponding to have spin degrees of freedom along the string. There exist five perturbatively ( $g_s \rightarrow 0$ ) inequivalent superstring theories that are consistent if the space-time dimension  $D = 10$ . They do not have any tachyon in the spectrum if one performs the GSO projection, are space-time supersymmetric and consistently unify gravity with gauge theories. They admit, however, non-perturbative solutions corresponding to  $p$ -dimensional extended objects, called p-branes. If we take them into account one can see that the five inequivalent theories are all related to each other and the underlying unifying theory, called M-theory, is an 11-dimensional theory that at low-energy reduces to 11-dimensional supergravity. In conclusion the long-time puzzle of the existence of five perturbatively inequivalent string theories in 10 dimensions has now disappeared because we have understood that at

the non-perturbative level they are all related to each other and part of a unique 11-dimensional M-theory.

We conclude this section by observing that string theory is not in contradiction with field theory but it is an extension of field theory as quantum mechanics and the theory of special relativity are an extension of respectively classical mechanics and the galilean mechanics in the sense that there is a limit corresponding to sending the Planck constant  $\hbar$  to zero and the speed of light  $c$  to  $\infty$  where we recover respectively classical mechanics and galilean mechanics. Analogously in the case of string theory one can recover field theory if we send the string tension to  $\infty$  or equivalently the Regge slope to zero ( $\alpha' \rightarrow 0$ ). In this way one goes from string theory containing quantum gravity to field theory containing only classical gravity.

### 3. Dp-BRANES

The massless bosonic spectrum of type II theories consists of a NS-NS sector with a graviton  $G_{\mu\nu}$ , a dilaton  $\phi$  and a two-form potential  $B_{\mu\nu}$  and of a R-R sector with a one-form  $C_1$  and a three-form potential  $C_3$  in the case of type IIA and with a scalar  $C_0$ , a two-form potential  $C_2$  and a four-form potential  $C_4$  in the case of type IIB. They contain also a massless fermionic sector consisting of two gravitinos and two dilatinos. A one-form potential as the electromagnetic field is coupled naturally with a point-particle through the well known coupling term  $\int C_\mu dx^\mu = \int C_1$ . So it is natural to think that a two-form potential is coupled to a string through the coupling  $\int \frac{1}{2} C_{\mu\nu} dx^\mu \wedge dx^\nu = \int C_2$ , and more generally a  $(p+1)$ -form potential is coupled to a  $p$ -dimensional object (called p-brane) through the interaction term  $\int C_{p+1}$ . In the previous formulas we have used for convenience the formalism of the forms where the form field is given by

$$C_{p+1} = \frac{1}{(p+1)!} C_{\mu_1 \dots \mu_{p+1}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{p+1}} \quad (18)$$

The low-energy string effective action for type II theories is given by

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G} \left\{ R - \frac{1}{2} G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \right. \\ \left. - \frac{1}{2 \cdot 3!} e^{-\phi} H_3^2 - \sum_p \frac{1}{2 \cdot (p+2)!} F_{p+2}^2 e^{(3-p)\phi/2} \right\} \quad (19)$$

where the ten-dimensional Newton constant is given by  $2\kappa^2 = (2\pi)^7 (\alpha')^4 g_s^2$ ,  $H_3$  is the field strength corresponding to the NS-NS two-form potential and  $F_{p+2}$  is the field strength corresponding to the potential  $C_{p+1}$ . The action in eq.(19) contains additional terms that are not important for our considerations.

Dp-branes are non-perturbative solutions of the classical equations of motion obtained from the previous low-energy string effective action in which at least one of the R-R fields is turned on. One starts from the following ansatz for a Dp-brane solution:

$$(ds)^2 \equiv G_{\mu\nu} dx^\mu dx^\nu = [H(r)]^{(p-7)/8} \eta_{\alpha\beta} dx^\alpha dx^\beta + \\ + [H(r)]^{(p+1)/8} \delta_{ij} dx^i dx^j \quad (20)$$

for the metric, where we have called the coordinates  $\mu, \nu$  with the indices  $(\alpha, \beta)$  for the directions along the world-volume of the brane and with the indices  $(i, j)$  for those transverse to the brane. In particular  $\alpha, \beta = 0, \dots, p$  and  $i, j = (p+1), \dots, 9$ . In addition to the metric we have also the dilaton and a R-R field turned on and they are given by:

$$e^{-(\phi-\phi_0)} = [H(r)]^{(p-3)/4}, \quad C_{01\dots p} = \frac{1}{H} - 1 \quad (21)$$

The harmonic function  $H$  is given by:

$$H(r) = 1 + \frac{K_p N}{r^{7-p}}, \quad r^2 = \sum_i x_i^2 \quad (22)$$

where

$$K_p = \frac{2\kappa^2 \tau_p}{(7-p)\Omega_{8-p}}, \quad \Omega_q = \frac{2\pi^{(q+1)/2}}{\Gamma(\frac{q+1}{2})} \quad (23)$$

and  $\Omega_q$  is the volume of the  $q$ -dimensional sphere. The tension of the brane, that is equal to its mass per unit of p-volume, is given by

$$\frac{M}{p\text{-volume}} = \int d^{9-p}x \theta_{00} = \tau_p N \quad (24)$$

while its charge is given by

$$\begin{aligned} & \frac{1}{\sqrt{2\kappa}} \int_{S^{8-p}} e^{-((p-3)\phi/2} F_{8-p} = \\ & = \sqrt{2\kappa} \tau_p N \equiv \mu_p N \end{aligned} \quad (25)$$

In terms of the string parameters  $g_s$  and  $\alpha'$  they are given by

$$\tau_p = \frac{(2\pi\sqrt{\alpha'})^{-p}}{g_s\sqrt{\alpha'}} \quad , \quad \mu_p = \sqrt{2\pi} (2\pi\sqrt{\alpha'})^{3-p} \quad (26)$$

In string theory the Dp-branes are characterized by the fact that open strings have their end-points attached to their world-volume. The spectrum of open strings having their end-points attached to the world-volume of a Dp-brane (i.e. satisfying Neumann boundary conditions on the directions along the world-volume of the brane and Dirichlet boundary conditions along the directions transverse to the brane) is given by the following formula:

$$\alpha' k_{||}^2 + \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n + \sum_t t \psi_t^\dagger \cdot \psi_t - a = 0 \quad (27)$$

where  $a = \frac{1}{2}[0]$  in the NS [R] sector and  $k_{||}$  is the momentum of the string parallel to the brane. In particular the massless states in the NS sector are given by  $(\psi_{1/2}^\alpha, \psi_{1/2}^i)|0, k >$  corresponding to a gauge boson  $A_\alpha$  and to  $(9-p)$  Higgs scalars  $\Phi^i$  related to the translational modes of the brane along the directions transverse to its world-volume. These gauge and scalar fields living on the world-volume of a Dp-brane become non-abelian transforming all of them according to the adjoint representation of the gauge group if instead of a single Dp-brane we have a bunch of  $N$  coincident Dp-branes. In this case in fact we get  $N^2$  massless states corresponding to the fact that the open strings can have their end-points on each of the  $N$  branes. In conclusion, while a single Dp-brane will have an abelian gauge field and  $(9-p)$  Higgs fields living on its world-volume, a bunch of  $N$  coincident Dp-branes will have a non-abelian gauge field and  $(9-p)$  Higgs fields all transforming according to the adjoint representation of  $U(N)$ .

The low-energy dynamics of a Dp-brane is described by the Born-Infeld action that is given by:

$$\begin{aligned} S_{BI} = & -\tau_p \int d^{p+1}x \, e^{-(3-p)\phi/4} \times \\ & \sqrt{-\det [G_{\alpha\beta} + e^{-\phi/2} (B_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta})]} + \\ & + \tau_p \int_{V_{p+1}} \sum_n C_n e^{(2\pi\alpha' F+B)} \end{aligned} \quad (28)$$

It contains the coordinates of the brane represented by the gauge field  $A_\alpha$  living on the brane with field strength  $F_{\alpha\beta}$  and by the transverse brane coordinates related to the Higgs fields by the relation  $x^i \equiv 2\pi\alpha'\Phi^i$ . They correspond to the massless open string excitations of the NS sector. This means that the dynamics of a brane is determined by the open strings having their end-points on the brane. The Born-Infeld action in eq.(28) contains also the bulk fields (i.e. fields living in the entire ten-dimensional space and not just on the brane) corresponding to the massless closed string excitations of the NS-NS and R-R sectors. What appears in eq.(28) is actually their pullback on the brane defined by

$$G_{\alpha\beta} = G_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu \quad , \quad B_{\alpha\beta} = B_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu \quad (29)$$

with a similar expression for the R-R fields. In the case of a system of  $N$  coincident branes the Born-Infeld action gets modified by the fact that the coordinates of the branes become non-abelian fields and the brane tension  $\tau_p$  gets multiplied with a factor  $N$ . The complete expression of the non-abelian Born-Infeld action is not yet known. But for our purpose it is sufficient to consider the non-abelian extension given in Ref. [3] where the symmetrized trace is introduced.

On the one hand a system of  $N$  Dp-branes is a classical solution of the low-energy string effective action whose low-energy dynamics is described by the Born-Infeld action. In particular it can be shown that a system of  $N$  coincident Dp-branes is a BPS state preserving  $1/2$  supersymmetry (corresponding to 16 preserved supersymmetries) and as a consequence they are not interacting. This can be easily seen by plugging the classical solution given in eq.s (20) and (21) in the Born-Infeld

action in eq.(28). In fact if we do that neglecting the coordinates of the brane we get

$$\tau_p \int d^{p+1}x \left\{ -H^{[(p-7)(p+1)-(p-3)^2]/16} + \frac{1}{H} - 1 \right\} = -\tau_p \int d^{p+1}x \quad (30)$$

that is independent on the distance  $r$  between the brane probe described by the Born-Infeld action and the system of  $N$  coincident branes described by the classical solution.

On the other hand a system of  $N$  Dp-branes has a  $U(N)$  gauge theory living on its world-volume with 16 supersymmetries corresponding, in the case of  $p = 3$ , to  $\mathcal{N} = 4$  super Yang-Mills in four dimensions that is a conformal invariant theory with vanishing  $\beta$ -function. Its Lagrangian can be obtained by expanding the first term of the Born-Infeld action up to the quadratic order in the gauge fields living on the brane. Neglecting the term independent from the gauge fields that we have already computed in eq.(30) we get the following Lagrangian:

$$L = \frac{1}{g_{YM}^2} \left[ -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + \frac{1}{2} \partial_\alpha \Phi^i \partial^\alpha \Phi^i \right] + \dots \quad (31)$$

where the gauge coupling constant is a constant given by:

$$g_{YM}^2 = \frac{2}{\tau_p (2\pi\alpha')^2} = \frac{2g_s \sqrt{\alpha'} (2\pi\sqrt{\alpha'})^p}{(2\pi\alpha')^2} \quad (32)$$

In particular for  $p = 3$  we get  $g_{YM}^2 = 4\pi g_s$ . The action in eq.(31) corresponds to the dimensional reduction of the  $\mathcal{N} = 1$  super Yang-Mills in ten dimensions to  $(p+1)$  dimensions.

The previous considerations imply that the low-energy dynamics of branes can be used to determine the properties of gauge theories and viceversa.

Let us analyze now what is the range of validity of the classical solution given in eq.s (20) and (21) restricting to the most interesting case of  $p = 3$ . In order to be able to use classical gravity we need, on the one hand, to neglect closed string loops and, on the other hand, to restrict the curvature of the solution to be small. The

first condition implies that  $g_s \ll 1$ . But, if we look at eq.(22) and we keep  $N$  small, the condition  $g_s \ll 1$  is equivalent to large values of  $r$  where the metric is almost Minkowski. Therefore in this regime the curvature is also small. This means that the classical solution provides a consistent description of the brane for large values of  $r$ . On the other hand, if we take the number of branes to be large in such a way that  $Ng_s$  is not necessarily small, then we can go to the near-horizon ( $r \rightarrow 0$ ) limit of the classical solution and we can ask ourselves: what is the value of  $Ng_s$  corresponding to a small curvature? It turns out that this happens for strong values of the 't Hooft coupling  $Ng_s \gg 1$ . This can formally be seen by taking the low-energy ( $\alpha' \rightarrow 0$ ) and the near-horizon ( $r \rightarrow 0$ ) limit of the D3-brane classical solution keeping the quantity  $U \equiv \frac{r}{\alpha'}$  fixed and corresponds to neglect the term 1 in eq.(22) as follows from

$$H = 1 + \frac{4\pi g_s (\alpha')^2 N}{r^4} = 1 + \frac{4\pi g_s N}{(\alpha')^2 U^4} \rightarrow \frac{4\pi g_s N}{(\alpha')^2 U^4} \quad (33)$$

In this limit we get the metric of  $AdS_5 \times S^5$ :

$$\frac{ds^2}{\alpha'} \rightarrow \frac{U^2}{b^2/\alpha'} dx_{3+1}^2 + \frac{b^2/\alpha'}{U^2} dU^2 + \frac{b^2}{\alpha'} d\Omega_5^2 \quad (34)$$

where the radii of  $AdS_5$  and of  $S^5$  are equal and given by

$$R_{AdS_5}^2 = R_{S^5}^2 = b^2 = \sqrt{4\pi g_s N} \alpha' \quad (35)$$

These formulas imply that the condition of small curvature requires that  $g_s N \gg 1$ . In conclusion the classical solution provides a good description of the brane at large distance ( $r \rightarrow \infty$ ) if  $Ng_s \ll 1$  and in the near-horizon limit ( $r \rightarrow 0$ ) if  $Ng_s \gg 1$ .

In the next section starting from the previous considerations we will formulate the Maldacena conjecture for  $\mathcal{N} = 4$  super Yang-Mills theory.

#### 4. MALDACENA CONJECTURE

In the previous section we have seen that a D3-brane is described at low-energy either by the

Born-Infeld action that for  $\alpha' \rightarrow 0$  reduces to  $\mathcal{N} = 4$  super Yang-Mills or by a classical solution of supergravity equations. They are different but equivalent ways of describing a D3-brane. In the following we will use these two different, but equivalent descriptions of a D3-brane for arriving at the Maldacena conjecture [4] stating that  $\mathcal{N} = 4$  super Yang-Mills is equivalent to ten-dimensional type IIB string theory compactified on  $AdS_5 \times S^5$ . In particular, if we consider the low-energy limit ( $\alpha' \rightarrow 0$ ) of the Born-Infeld action, it consists of a brane action corresponding to that of four-dimensional  $\mathcal{N} = 4$  super Yang-Mills and by a term describing the interaction of the brane with the bulk fields. However, in the limit of  $\alpha' \rightarrow 0$  the interaction term, being proportional to  $\kappa \sim (\alpha')^2$ , is vanishing. In this limit also the bulk fields are not interacting. Therefore from the point of view of the Born-Infeld action in the low-energy limit we get  $\mathcal{N} = 4$  super Yang-Mills plus free gravitons or more in general free bulk fields.

On the other hand, if we look at the classical solution given in eq.(20) we see that it interpolates between flat Minkowski space ( $r \rightarrow \infty$ ) and a long throat in the near-horizon limit ( $r \rightarrow 0$ ). If we have sufficiently soft gravitons (i.e. gravitons with wave length much bigger than the radius of the throat  $b$ ) outside the throat they cannot interact with the excitations far down in the throat as it is confirmed by the fact that their absorption cross-section is vanishing at low energy [5,6]. On the other hand a string excitation far down inside the throat, although its proper energy (the energy measured in the reference frame instantaneously at rest at  $r$ ) diverges at low energy ( $\alpha' \rightarrow 0$ ), being proportional to  $E_p \sim 1/\sqrt{\alpha'}$ , is not negligible because its energy measured in the frame of reference where the time is the one appearing in the first term of the r.h.s. of eq.(34) is given by:

$$E_t \sim \frac{r}{b} E_p \sim \frac{r}{b\sqrt{\alpha'}} \sim \frac{r}{\alpha'} = U \quad (36)$$

that is kept fixed in the limit  $\alpha' \rightarrow 0$ . Therefore from the point of view of the classical solution we are left with free gravitons and all the string excitations living far down inside the throat that are described by type IIB string theory compactified

on  $AdS_5 \times S^5$ .

By comparing this result with the one obtained from the Born-Infeld action Maldacena has formulated the conjecture that  $\mathcal{N} = 4$  super Yang-Mills is equivalent to type IIB string theory compactified on  $AdS_5 \times S^5$ . The precise relation between the parameters of the gauge and string theories is given in eq.(35), where  $N$  is equal to the number of colours in the gauge theory and to the flux of the 5-form field strength in the supergravity solution. Since the classical solution in eq.(34) is a good approximation when the radii of  $AdS_5$  and  $S^5$  are very big

$$\frac{b^2}{\alpha'} \gg 1 \implies N g_{YM}^2 \equiv \lambda \gg 1, \quad (37)$$

in the strong coupling limit of the gauge theory we can restrict ourselves to the type IIB supergravity compactified on  $AdS_5 \otimes S^5$ .

In conclusion, according to the Maldacena conjecture, classical supergravity is a good approximation if  $\lambda \gg 1$ , while in the 't Hooft limit in which  $\lambda$  is kept fixed for  $N \rightarrow \infty$  classical string theory is a good approximation for  $\mathcal{N} = 4$  super Yang-Mills. In the 't Hooft limit in fact string loop corrections are negligible ( $g_s \ll 1$ ) as it follows from the equation:  $\lambda = 4\pi g_s N$  for  $\lambda$  fixed and  $N \rightarrow \infty$ . Finally Yang-Mills perturbation theory is a good approximation when  $\lambda \ll 1$ .

The strongest evidence for the validity of the Maldacena conjecture comes from the fact that both  $\mathcal{N} = 4$  super Yang-Mills and type IIB string compactified on  $AdS_5 \otimes S^5$  have the same symmetries. They are, in fact, both invariant under 32 supersymmetries, under the conformal group  $O(4, 2)$ , corresponding to the isometries of  $AdS_5$ , under the  $R$ -symmetry group  $SU(4)$ , corresponding to the isometries of  $S^5$  and under the Montonen-Olive duality [7] based on the group  $SL(2, Z)$ .

The validity of the Maldacena conjecture has by now been confirmed by many checks and this is the first time that a string theory is recognized to come out from a gauge theory. In particular it is important to stress that this does not contradict the fact that a string theory contains gravity while the gauge theory does not, because in this case the two theories live in different spaces: IIB

string theory lives on  $AdS_5 \otimes S_5$ , while  $\mathcal{N} = 4$  super Yang-Mills lives in our four-dimensional Minkowski space. A new puzzle, however, arises in this case because we usually connect a string theory with a confining gauge theory, while instead  $\mathcal{N} = 4$  super Yang-Mills is a conformal invariant theory and therefore is in the Coulomb and not in the confining phase.

## 5. NONCONFORMAL GAUGE THEORIES

In the previous section we have seen that IIB string theory compactified on  $AdS_5 \times S^5$  (IIB supergravity for large values of the 't Hooft coupling) can be used to study the properties of  $\mathcal{N} = 4$  super Yang-Mills. It is desirable to extend the previous analysis to less supersymmetric and non conformal gauge theories. Many different attempts have been tried to describe non-conformal and less supersymmetric theories using supergravity solutions. In this section we limit ourselves to discuss how to use the fractional branes for studying the properties of  $\mathcal{N} = 2$  super Yang-Mills.

Let us consider type IIB string theory on the background  $R^{1,5} \times R^4/Z_2$  consisting of 6-dimensional Minkowski space times the orbifold  $R^4/Z_2$  where  $Z_2$  acts on the orbifolded components by changing their sign:  $x^i \rightarrow -x^i$  where  $i = 6, 7, 8, 9$ . Such a background breaks 1/2 supersymmetry with respect to ten-dimensional Minkowski space and on the world-volume of a D3-brane that breaks an additional 1/2 supersymmetry, one gets a supersymmetric theory with 8 instead of 16 charges as in the case of  $\mathcal{N} = 4$  super Yang-Mills. On the other hand in the case of an orbifold one has a more general type of branes, called fractional Dp-branes. They are characterized by the fact that they are stuck at the orbifold fixed point ( $x^6 = x^7 = x^8 = x^9 = 0$ ), have a charge and tension that are a fraction of those of a normal brane (in the case of a  $Z_2$  orbifold this fraction is just 1/2) and can be seen as D(p+2)-branes wrapped on vanishing exceptional cycles located at the orbifold fixed points.

Since we want to study four-dimensional gauge theories let us consider a fractional D3-brane of

the orbifold  $R^4/Z_2$ , let us write its world-volume action and let us use it to determine the corresponding IIB supergravity classical solution. Such a D3-brane is coupled to the untwisted bulk fields corresponding to the metric  $G_{\mu\nu}$  and the 4-form potential  $C_4$  as a normal D3-brane. In addition it is coupled to the two twisted fields  $b, c$  obtained by wrapping the two 2-form potentials  $B_2$  and  $C_2$  on the vanishing exceptional cycle:

$$B_2 = b\omega_2, \quad C_2 = c\omega_2 \quad (38)$$

A fractional D3-brane is described by the following Born-Infeld action [8]:

$$S_{BI} = \frac{\tau_3}{4\pi^2\alpha'} \left[ - \int d^4x \, b \times \sqrt{-\det(G_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta})} + \int C_4 b + \int \mathcal{A}_4 \right] + \frac{c}{2\pi g_s} \left( \frac{1}{32\pi^2} \int d^4x F_{\alpha\beta} \tilde{F}^{\alpha\beta} \right) \quad (39)$$

where  $\mathcal{A}_4$  is the Hodge dual of  $c$ .

The classical supergravity solution corresponding to a system of  $M$  fractional D3-branes is given by the following expressions for the untwisted fields:

$$ds^2 = H^{-1/2} \eta_{\alpha\beta} dx^\alpha dx^\beta + H^{1/2} \delta_{ij} dx^i dx^j \quad (40)$$

and

$$\tilde{F}_5 = H^{-1} dx^0 \dots dx^3 + *d(H^{-1} dx^0 \dots dx^3) \quad (41)$$

where the function  $H$  satisfies the equation [8,9]:

$$-\partial_i \partial^i H(r, \rho) = 2\kappa^2 \tau_3 M \delta(x^4) \dots \delta(x^9) + \frac{(4\pi\alpha' g_s M)^2}{\rho^2} \delta(x^6) \dots \delta(x^9) \quad (42)$$

with  $r^2 = \sum_{i=4}^9 x_i^2$  and  $\rho^2 = (x^4)^2 + (x^5)^2$ .

The twisted fields are only a function of the coordinates  $x^4$  and  $x^5$  and are given by [8,9]

$$\gamma(z) \equiv c + ib = 2i\pi\alpha' g_s \left[ \frac{\pi}{g_s} + 2M \log \frac{z}{\epsilon} \right] \quad (43)$$

that implies

$$b = \frac{(2\pi\sqrt{\alpha'})^2}{2} + 4\pi\alpha' M g_s \log \frac{\rho}{\epsilon} \quad (44)$$



and

$$c = -4\pi\alpha' M g_s \theta \quad , \quad z \equiv x^4 + ix^5 = \rho e^{i\theta} \quad (45)$$

It can be seen that the metric has a naked singularity of the repulson type (the gravitational force goes to zero and becomes repulsive) at short distance. On the other hand if we plug the classical solution given above in the Born-Infeld action of a probe fractional D3-brane we see that its tension becomes negative at a distance (called enhançon) bigger than the one corresponding to the repulson singularity. This means that the classical solution makes sense only for distances bigger than the enhançon. When we insert the classical solution in the probe action we find that the gauge theory living on the brane has a running coupling constant given by:

$$\frac{1}{g_{YM}^2(\rho)} = \frac{1}{g_{YM}^2(\epsilon)} + \frac{M}{4\pi^2} \log \frac{\rho}{\epsilon} \quad (46)$$

and a  $\theta_{YM}$  angle given by:

$$\theta_{YM} = 2M\theta \quad , \quad g_{YM}^2(\epsilon) \equiv 8\pi g_s \quad (47)$$

where  $\rho$  that originally is the distance between the brane probe and the branes that generate the classical solution becomes the renormalization group scale of the gauge theory living on the brane. The previous analysis provides a geometrical interpretation of the value of  $\mu$  where the gauge coupling constant becomes infinite corresponding in gauge theory to  $\Lambda_{QCD}$  (the dimensional constant generated by dimensional transmutation in gauge theories) and to the enhançon (scale where the brane probe becomes tensionless) in the brane dynamics. From eq.(46) one can compute the  $\beta$ -function of the gauge theory that is given by [8]:

$$\beta(g_{YM}) = -\frac{M g_{YM}^3}{8\pi^2} \quad (48)$$

and that is the  $\beta$ -function of  $\mathcal{N} = 2$  super Yang-Mills. We see that the classical solution describes the perturbative properties of  $\mathcal{N} = 2$  super Yang-Mills, but fails to reproduce the non-perturbative instanton contribution.

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